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RITZ VARIATIONAL METHOD FOR THE FREE HARMONIC VIBRATION SOLUTIONS OF SLENDER BEAMS ON TWO-PARAMETER ELASTIC FOUNDATIONS

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Abstract

This study presents Ritz variational method for the free transverse harmonic vibration solutions of slender beams on two-parameter elastic foundations (SBo2PEFs). The studied problem is a soil-structure interaction problem of dynamics that is important in the dynamic design of foundations and buried pipelines. The domain equation is derived using variational calculus, and the total energy functional was found for harmonic vibrations in terms of the modal displacement W(x) and the derivatives $W'(x), W''(x)$. *Minimization criteria with respect to the generalized parameter of the displacement is used to find the characteristic frequency equation. The obtained Ritz equation is an eigenvalue problem. It was found that for simply supported SBo2PEF, the exact sinusoidal shape function used gave the exact eigenfrequency for any mode of vibration. For clamped-clamped SBo2PEF, a one-parameter shape function gave accurate fundamental frequency. For cantilever SBo2PEF, a one-parameter shape function gave accurate fundamental frequency solutions.*

1.0 INTRODUCTION

Beams supported on elastic foundations (BoEF) have many important applications in civil, mechanical engineering and represent a complex soil-structure interaction problem [1 – 2]. Due to the mathematically rigorous involvements, a limited number of analytical solutions have been found for BoEF. The problems of BoEF have been modeled using variational techniques and equilibrium methods. In the modeling process, beam and foundation models are integrated in seeking the vibration equations.

Euler and Bernoulli developed Euler-Bernoulli beam theory (EBT) as a simple version of linear theory of elasticity by using assumption that cross-sectional planes are perpendicular to middle plane. The EBT disregards transverse shear strains and hence is inapplicable to moderately thick beams and thick beams having shearing deformations influence their structural behaviour. Hence EBT is limited to slender beams. Other beam theories were developed for thick beams and moderately thick beams by Timoshenko, Ike [3] and Sayyad and Ghugal [4] to incorporate transverse shear strains.

The foundation model is also a complex problem and determines the governing equation of the soil-

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structure interaction problem. Winkler foundation represents a simple idealization of a foundation. The Winkler model depicted in Figure 1 represents the foundation as a mechanical analogue of proximately packed vertical springs that are mutually independent, and linearly elastic.

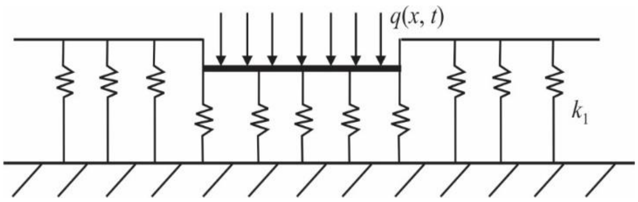

Figure 1: Analogue of Winkler foundation idealization

Winkler foundation however is unable to account for shear deformation resulting in discontinuous issues [5]. The foundation reaction, $r_s(x, t)$, at time t, is given by:

$$r_s(x,t) = k_1 w(x,t) \tag{1}$$

Where, k_1 is the Winkler model parameter which can be constant or variable [6]; and $w(x, t)$ denotes deflection at any time, t.

Improvements to Winkler model were made by Hetenyi, Filonenko-Borodich (FB), Pasternak and Vlasov by introducing coupling parameters. Such coupling parameters represent the interacting effects and define a second parameter, k_2, in the model [7]. An analogue of the two-parameter foundation mode is depicted in Figure 2.

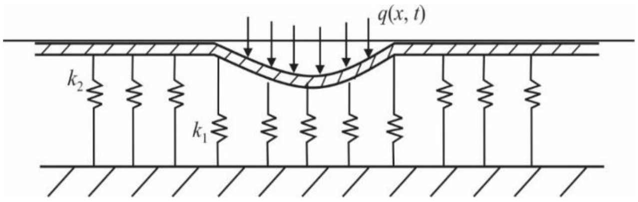

Figure 2: Analogue of two-parameter elastic foundation (2PEF)

The reaction in 2PEF has been expressed thus [5]:

$$r(x,t) = k_1 w(x,t) - k_2 \frac{\partial^2 w(x,t)}{\partial x^2} \tag{2}$$

This work assumes a slender beam on two-parameter elastic foundation (SBo2PEF). Several researchers have studied the vibration of BoEFs. Their studies used various methods including finite element method (FEM), variational iteration methods (VIMs), Adomian decomposition methods (ADMs), Rayleigh-Ritz method, generalized integral transformation methods (GITMs), integral transform methods (ITMs) and analytical methods.

Rao and Raju [8] derived analytical eigensolutions for the natural transverse vibration solutions for thin beams on Pasternak foundations, and developed exact solutions for the first five modes of vibration for SBo2PEF for a variety of end support conditions. Rahbar-Ranji and Shahbaztabar [9] utilized Legendre polynomials and Rayleigh Ritz method to derive natural transverse vibration eigensolutions of SBo2PEF.

Franciosi and Masi [10] developed accurate eigenfrequencies for the natural vibrations of SBo2PEF for various end support conditions. Naidu and Rao [11] used the FEM to study the “vibrations of initially stressed uniform beams on two-parameter elastic foundation.”

Zhou [12] developed “a general solution to vibrations of beams resting on variable Winkler foundation”. Cincin and Coskun [13] and Cincin and Coskun [14] explored the use of ADM for the analysis of vibrating beams on partially elastic foundation.

Tazabekova et al [15] utilized VIM for natural vibration frequency determinations of BoEF, and found accurate solutions. Al-Azzawi and Daud [16] studied the “free vibration analysis of non-prismatic beam on variable Winkler elastic foundation”.

Finite element methodology (FEM) using stress technique was implemented in flexural analysis of BoEF by Wieckowski and Swiatkiewicz [17], but their study failed to consider vibration analysis. Kanwal et al [18] studied vibration analysis of different models of BoEFs. Ogunbamike [19] studied the vibration of SBo2PEF with focus on the FB foundation model. “Numerical free vibration analysis of higher-order shear deformable beams resting on two-parameter elastic foundation” was studied by [20]. “Forced vibration: Axially functionally graded thin beams on elastic foundation” was studied in by [21]. “Vibration analysis of a beam on a nonlinear elastic foundation” was studied by [22]. Sahin [23] investigated “Vibration of a composite elastic beam on an inhomogeneous elastic foundation.” Ike [24] studied “Sumudu transform method for finding the transverse natural harmonic vibration frequencies of Euler-Bernoulli beams” but did not consider elastic foundation interactions. Ike [25] studied “Ritz variational method for buckling analysis of Euler-Bernoulli beams resting on two-parameter foundations” but did not consider vibration analysis.

Ike [26] studied "Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation" but did not consider two-parameter foundations. Ike [27] studied "Free vibration of thin beams on Winkler foundations using generalized integral transform method" but did not consider two-parameter foundations.

Recently, Ike [28] used variational methods for thick beam vibration analysis based on sinusoidal shear deformable modeling. The resulting equations were solved using finite sine transform method for the case of simply supported boundary conditions. Ike [29] used Stodola-Vianello iteration method in an innovative way for buckling solutions of bisymmetric beams, but did not consider free vibration analysis. Ike [30] used an equilibrium method for the derivation of first-order shear deformable thick beam on two-parameter elastic foundation and subjected to static loads. Closed form static flexure solutions were obtained by integrating the domain equations using Naviers series method for simply supported BCs. Ike [30] didn't however considered free vibration analysis.

In this work, Ritz variational method (RVM) is utilized for the harmonic transverse vibration analysis of SBo2PEF. The RVM is adopted in the study because it is amenable to numerical computations and can enable the development of close approximations to the exact solutions of the beam on elastic foundation problem. Another reason is that exact mathematically rigorous methods of solution involve hyperbolic functions and transcendental functions in the characteristic frequency equation which are difficult to solve; the RVM offers simplifications to the solution process and yet gives accurate eigensolutions.

2.0 METHODOLOGY

2.1 Variational Formulation

The basic assumptions are:

(i) The beam material is composed of homogeneous material with linear elastic and isotropic properties.
(ii) The deflections remain so infinitesimal compared with beam depth, so that small infinitesimal elasticity theory is used.
(iii) The elastic foundation parameters are non-variable.
(iv) The beam is slender, and thin beam theory is used.
(v) The equation relating stress and strain as Hooke's law is one-dimensional.

Displacement components are:

$$u(x,y,z,t) = u(x,t) = -z\frac{\partial w(x,t)}{\partial x}; \quad v(x,y,z) = v(x,t) = 0; \qquad (3)$$

$$w(x, y=0, z=0, t) = w(x,t)$$

Where, x is the longitudinal coordinate axis, u, v, w denote the x, y, z displacement components respectively.

The non-zero strain is the axial strain in x direction, ε_{xx}, and it is:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad (4)$$

The non-vanishing axial stress (σ_{xx}) is, from the one-dimensional constitutive law, given by:

$$\sigma_{xx} = E\varepsilon_{xx} = -Ez\frac{\partial^2 w}{\partial x^2} \qquad (5)$$

Strain energy of the thin beam, U_b is

$$U_b = \frac{1}{2}\int_0^l\iint_A \sigma_{xx}\varepsilon_{xx}\,dxdydz = \frac{1}{2}\int_0^l\iint_A E\varepsilon_{xx}^2\,dxdA = \frac{1}{2}\int_0^l\iint_A\left(-z\frac{\partial^2 w}{\partial x^2}\right)^2 EdxdA \qquad (6)$$

$$U_b = \frac{1}{2}\int_0^l\iint_A z^2 dA\left(\frac{\partial^2 w}{\partial x^2}\right)^2 Edx = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \qquad (7)$$

The strain energy of the two-parameter elastic foundation 2PEF is

$$U_f = \frac{1}{2}\int_{-b/2}^{b/2}\int_0^l r_{s_1} w(x,t)\,dxdy + \frac{1}{2}\int_{-b/2}^{b/2}\int_0^l r_{s_2} w'(x)\,dxdy \qquad (8)$$

r_{s1} and r_{s2} are reactions from the elastic foundation, b is the beam width.

$$r_{s_1} = k_1 w(x,t); \quad r_{s_2} = k_2 w'(x,t) \qquad (9)$$

k_1 and k_2 are the stiffnesses of the two-parameter models.

$$U_f = \frac{1}{2}\int_0^l bk_1 w^2 dx + \frac{1}{2}\int_0^l bk_2\left(w'(x,t)\right)^2 dx = \frac{1}{2}\int_0^l \bar{k}_1 w^2 dx + \frac{1}{2}\int_0^l \bar{k}_2 (w')^2 dx \qquad (10)$$

Where, $\bar{k}_1 = bk_1, \quad \bar{k}_2 = bk_2 \qquad (11)$

The kinetic energy, T is:

$$T = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial w}{\partial t}\right)^2 dx \qquad (12)$$

ρ is beam material density, A is area of the cross section.

For harmonic vibrations, $w(x,t) = W(x)\sin(\omega_n t - \phi)$ (13)

$W(x)$ denotes modal deflection, ω_n denotes natural frequency, ϕ denotes phase.

Total energy (E_t) expression is then,

$$E_t = \frac{1}{2}\int_0^l \left(EI\left(W''(x)\right)^2 + k_1\left(W(x)\right)^2 + k_2\left(W'(x)\right)^2 - \rho A\omega_n^2\left(W(x)\right)^2\right)dx$$

(14)

2.2 Ritz Variational Method (RVM)

The Ritz variational method (RVM) seeks to minimize E_t with respect to the unknown generalized displacement parameters of $W(x)$ in order to achieve dynamic equilibrium. Thus if:

$$W(x) = \sum_{n=1}^{\infty} c_n\varphi_n(x) \qquad (15)$$

$n = 1, 2, 3, \ldots$

Where, $\varphi(x)$ are shape functions which satisfy boundary conditions, c_n are generalized parameters.

Then E_t is expressed in terms of c_n and $\varphi_n(x)$ as:

$$E_t = \frac{1}{2}\int_0^l \left\{EI\left(\sum_{n=1}^{\infty} c_n\varphi_n''(x)\right)^2 + k_1\left(\sum_{n=1}^{\infty} c_n\varphi_n(x)\right)^2 + k_2\left(\sum_{n=1}^{\infty} c_n\varphi_n'(x)\right)^2 - \rho A\omega_n^2\left(\sum_{n=1}^{\infty} c_n\varphi_n(x)\right)^2\right\}dx \qquad (16)$$

Approximate solutions are obtained using a finite N degree of freedom representation for $W(x)$ as:

$$W(x) = \sum_{n=1}^{N} c_n\varphi_n(x) \qquad (17)$$

Then,

$$E_t = \frac{1}{2}\sum_{n=1}^{N} c_n^2\int_0^l \left(EI\left(\varphi_n''(x)\right)^2 + k_1\left(\varphi_n(x)\right)^2 + k_2\left(\varphi_n'(x)\right)^2 - \rho A\omega_n^2\left(\varphi_n(x)\right)^2\right)dx \qquad (18)$$

The criteria for extremizing E_t is:

$$\frac{\partial E_t}{\partial c_n} = 0 \qquad (19)$$

$n = 1, 2, 3, \ldots, N$

The resulting Ritz equations become:

$$\int_0^l \left(EI\varphi_n''(x)\varphi_{\bar{n}}''(x) + k_1\varphi_n(x)\varphi_{\bar{n}}(x) + k_2\varphi_n'(x)\varphi_{\bar{n}}'(x) - \rho A\omega_n^2\varphi_n(x)\varphi_{\bar{n}}(x)\right)dx = 0$$

(20)

$n = 1, 2, 3, \ldots, N;\ \bar{n} = 1, 2, 3, \ldots, N.$

3.0 RESULTS AND DISCUSSION

3.1 Simply Supported SBo2PEF

For simply supported SBo2PEF shown in Figure 3, the shape function $\varphi_n(x)$ should satisfy the equations:

$$\varphi_n(0) = \varphi_n(l) = \varphi_n''(0) = \varphi_n''(l) = 0 \qquad (21)$$

Suitable sinusoidal shape function for the boundary conditions (BCs) is:

$$\varphi_n(x) = \sin\frac{n\pi x}{l} \qquad (22)$$

Figure 3: Simply supported SBo2PEF

The Ritz equations become:

$$\int_0^l \left\{EI\left(\frac{n\pi}{l}\right)^2\left(\frac{\bar{n}\pi}{l}\right)^2 \sin\frac{n\pi x}{l}\sin\frac{\bar{n}\pi x}{l} + k_1\sin\frac{n\pi x}{l}\sin\frac{\bar{n}\pi x}{l} + k_2\left(\frac{n\pi}{l}\right)\left(\frac{\bar{n}\pi}{l}\right)\cos\frac{n\pi x}{l}\cos\frac{\bar{n}\pi x}{l} - \rho A\omega_n^2\sin\frac{n\pi x}{l}\sin\frac{\bar{n}\pi x}{l}\right\}dx = 0 \qquad (23)$$

Considering orthogonality properties of the shape function, the equation reduces to

$$\int_0^l \left(EI\left(\frac{n\pi}{l}\right)^4 \sin^2\frac{n\pi x}{l} + k_1\sin^2\frac{n\pi x}{l} + k_2\left(\frac{n\pi}{l}\right)^2\cos^2\frac{n\pi x}{l} - \rho A\omega_n^2\sin^2\frac{n\pi x}{l}\right)dx = 0$$

(24)

Simplifying, the frequency equation becomes:

$$EI\left(\frac{n\pi}{l}\right)^4 I_1 + k_1 I_1 + k_2\left(\frac{n\pi}{l}\right)^2 I_2 - \rho A\omega_n^2 I_1 = 0 \qquad (25)$$

$$I_1 = \int_0^l \sin^2\frac{n\pi x}{l}dx = \quad I_2 = \int_0^l \cos^2\frac{n\pi x}{l}dx \qquad (26)$$

The frequency equation is:

$$\rho A\omega_n^2 = EI\left(\frac{n\pi}{l}\right)^4 + k_1 + k_2\left(\frac{n\pi}{l}\right)^2 \qquad (27)$$

$$\omega_n^2 = \frac{1}{\rho A}\left(EI\left(\frac{n\pi}{l}\right)^4 + k_1 + k_2\left(\frac{n\pi}{l}\right)^2\right) = \frac{EI}{\rho A l^4}\left((n\pi)^4 + \alpha_1 l^4 + \alpha_2 l^2 (n\pi)^2\right)$$

(28)

Where, $\alpha_1 = \frac{k_1}{EI}, \quad \alpha_2 = \frac{k_2}{EI}$ (29)

$$\omega_n = \sqrt{\frac{EI}{\rho A l^4}}\left((n\pi)^4 + \alpha_1 l^4 + \alpha_2 l^2 (n\pi)^2\right)^{1/2} = \frac{\lambda_n^2}{l^2}\sqrt{\frac{EI}{\rho A}} \qquad (30)$$

$$\lambda_n^2 = \left((n\pi)^4 + \alpha_1 l^4 + \alpha_2 l^2 (n\pi)^2\right)^{1/2} \qquad (31)$$

The natural frequency parameters are computed for the first mode and for $\alpha_1 l^4 = 0, 1, 10, 100, 1000, 10,000$ and $\frac{\alpha_2 l^2}{\pi^2} = 0,\ 0.5, 1.0,\ 2.5;$ and presented in Table 1 along with previous Rao and Raju [8] solutions and solutions by Naidu and Rao [11]. Table 1 illustrates that the present result gave identical frequency parameters with previous solutions by [8], [9] and [11].

Table 1: Frequency parameters for simply supported SBo2PEF

Foundation parameters	Present	Rao and Raju [8]	FEM Naidu and Rao [11]	Rahbar-Ranji and Shahbaztabar [9]

$\alpha_1 l^4$	$\frac{\alpha_2 l^2}{\pi^2}$				
0	0	3.1416	3.1415	3.1415	
	0.5	3.4767	3.4767	3.4767	
	1.0	3.7360	3.7360	3.7360	
	2.5	4.2970	4.2970	4.2970	
1	0	3.1496	3.1496	3.1496	3.1496
	0.5	3.4827	3.4826	3.4826	3.4827
	1.0	3.7408	3.7407	3.7407	3.7408
	2.5	4.3002	4.3001	4.3001	4.3002
10	0	3.2193			3.2193
	0.5	3.5347			3.5347
	1.0	3.7831			3.7831
	2.5	4.3282			4.3282
100	0	3.7484	3.7483	3.7483	3.7484
	0.5	3.9608	3.9608	3.9608	3.9608
	1.0	4.1437	4.1437	4.1437	4.1437
	2.5	4.5824	4.5824	4.5824	4.5824
1000	0	5.7556			5.7556
	0.5	5.8184			5.8184
	1.0	5.8793			5.8793
	2.5	6.0513			6.0513
10,000	0	10.0243	10.024	10.024	10.0243
	0.5	10.0363	10.036	10.036	10.0363
	1.0	10.0483	10.048	10.048	10.0483
	2.5	10.0842	10.084	10.084	10.0842

3.2 SBo2PEF with Clamped-Clamped Ends

A SBo2PEF with both ends clamped is illustrated in Figure 4.

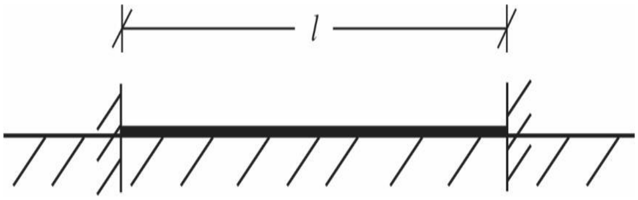

Figure 4: SBo2PEF clamped at both left and right ends

The required constraint equations for $\varphi(x)$ can be expressed as:

$$\varphi(0) = \varphi'(0) = \varphi(l) = \varphi'(l) = 0 \quad (32)$$

A one-parameter shape function is:

$$\varphi(x) = 1 - \cos\frac{2\pi x}{l} \quad (33)$$

The Ritz characteristic frequency equation is:

$$\int_0^l \left\{ EI\left(\left(\frac{2\pi}{l}\right)^2 \cos\frac{2\pi x}{l}\right)^2 + k_2\left(\frac{2\pi}{l}\sin\frac{2\pi x}{l}\right)^2 + k_1\left(1-\cos\frac{2\pi x}{l}\right)^2 - \rho A\omega_n^2\left(1-\cos\frac{2\pi x}{l}\right)^2 \right\} dx = 0 \quad (34)$$

$$EI\left(\frac{2\pi}{l}\right)^4 I_3 + k_2\left(\frac{2\pi}{l}\right)^2 I_4 + k_1 I_5 - \rho A\omega_n^2 I_5 = 0 \quad (35)$$

$$I_3 = \int_0^l \cos^2\frac{2\pi x}{l}dx = \frac{l}{2};\ \ I_4 = \int_0^l \sin^2\frac{2\pi x}{l}dx = \frac{l}{2};\ \ I_5 = \int_0^l \left(1-\cos\frac{2\pi x}{l}\right)^2 dx = \frac{3l}{2} \quad (36)$$

Thus,

$$\rho A\omega_1^2\frac{3l}{2} = EI\left(\frac{2\pi}{l}\right)^4\frac{l}{2} + k_2\left(\frac{2\pi}{l}\right)^2\frac{l}{2} + k_1\frac{3l}{2} \quad (37)$$

$$\omega_1^2 = \frac{1}{\rho A}\frac{2}{3}\left(\frac{EI}{2}\left(\frac{2\pi}{l}\right)^4 + \frac{k_2}{2}\left(\frac{2\pi}{l}\right)^2 + \frac{3k_1}{2}\right) = \frac{1}{\rho A}\left(\frac{16\pi^4}{3l^4}EI + \frac{4\pi^2 k_2}{3l^2} + k_1\right) \quad (38)$$

$$\omega_1^2 = \frac{EI}{\rho A l^4}\left(\frac{16}{3}\pi^4 + \frac{4\pi^2}{3}\frac{k_2 l^2}{EI} + \frac{k_1 l^4}{EI}\right) = \frac{EI}{\rho A l^4}\left(\alpha_1 l^4 + \frac{1}{3}\left((2\pi)^2\alpha_2 l^2 + (2\pi)^4\right)\right) \quad (39)$$

$$\omega_1 = \sqrt{\frac{EI}{\rho A}}\frac{\lambda_1^2}{l^2} \quad (40)$$

$$\lambda_1^2 = \left(\alpha_1 l^4 + \frac{1}{3}\left((2\pi)^2\alpha_2 l^2 + (2\pi)^4\right)\right)^{1/2} \quad (41)$$

The fundamental natural frequency parameters λ_1 are computed for SBo2PEF with clamped-clamped ends using Equation (41) and shown in Table 2. Table 2 also presents previous solutions by Naidu and Rao [11], and Rao and Raju [8], and confirms that present solutions are identical with previous solutions by [8] and less than 1.83% different from solutions by [11].

Table 2: Fundamental natural frequency parameter of a SBo2PEF with clamped-clamped ends

Foundation parameters	Present	Rao and Raju [8]	Naidu and Rao [11]

$\alpha_1 l^4$	$\frac{\alpha_2 l^2}{\pi^2}$			
0	0	4.7742	4.7742	4.7300 (-0.93%)
0	0.5	4.9168	4.9168	4.8669 (-1.01%)
0	1.0	5.0481	5.0481	4.9925 (-1.10%)
0	2.5	5.3903	5.3903	5.3183 (-1.34%)
1	0	4.7765	4.7765	4.7324 (-0.92%)
1	0.5	4.9190	4.9190	4.8691 (-1.01%)
1	1.0	5.0500	5.0500	4.9945 (-1.10%)
1	2.5	5.3913	5.3913	5.3200 (-1.32%)
100	0	4.9890	4.9890	4.9504 (-0.77%)
100	0.5	5.1649	5.1649	5.0706 (-1.83%)
100	1.0	5.2321	5.2321	5.1823 (-0.95%)
100	2.5	5.5433	5.5433	5.4773 (-1.19%)
1000	0	10.127	10.127	10.122 (-0.05%)
1000	0.5	10.143	10.143	10.137 (-0.06%)
1000	1.0	10.158	10.158	10.151 (-0.07%)
1000	2.5	10.247	10.247	10.194 (-0.52%)

The numbers in bracket represent percentage difference of present solutions and previous solutions by [11].

3.3 Cantilever SBo2PEF

A one parameter shape function that satisfies the boundary conditions of cantilever SBo2PEF shown in Figure 5 is:

Figure 5: Cantilever SBo2PEF

$$\varphi(x) = \left(1 - \cos\frac{\pi x}{2l}\right) \tag{42}$$

The Ritz characteristic frequency equation becomes:

$$\int_0^l \left\{ EI\left(\left(\frac{\pi}{2l}\right)^2 \cos\frac{\pi x}{2l}\right)^2 + k_1\left(1-\cos\frac{\pi x}{2l}\right)^2 + k_2\left(\frac{\pi}{2l}\sin\frac{\pi x}{2l}\right)^2 - \rho A\omega_n^2\left(1-\cos\frac{\pi x}{2l}\right)^2 \right\} dx = 0 \tag{43}$$

Simplifying,

$$EI\left(\frac{\pi}{2l}\right)^4 I_6 + k_1 I_7 + k_2\left(\frac{\pi}{2l}\right)^2 I_8 - \rho A\omega_n^2 I_7 = 0 \tag{44}$$

$$I_6 = \int_0^l \cos^2\left(\frac{\pi x}{2l}\right)dx = \frac{l}{2};\ I_7 = \int_0^l \left(1-\cos\frac{\pi x}{2l}\right)^2 dx = \frac{(3\pi-8)l}{2\pi}; \tag{45}$$

$$I_8 = \int_0^l \sin^2\left(\frac{\pi x}{2l}\right)dx = \frac{l}{2}$$

Then, the frequency equation is:

$$\rho A\omega_n^2 I_7 = EI\left(\frac{\pi}{2l}\right)^4 I_6 + k_1 I_7 + k_2\left(\frac{\pi}{2l}\right)^2 I_8 \tag{46}$$

Substituting for I_6, I_7, I_8 gives:

$$\rho A\omega_n^2\frac{(3\pi-8)l}{2\pi} = \left(EI\left(\frac{\pi}{2l}\right)^4\frac{l}{2} + k_1\frac{(3\pi-8)l}{2\pi} + k_2\left(\frac{\pi}{2l}\right)^2\frac{l}{2}\right) \tag{47}$$

$$\omega_n^2 = \frac{2\pi}{\rho A(3\pi-8)l}\left(EI\left(\frac{\pi}{2l}\right)^4\frac{l}{2} + k_1\frac{(3\pi-8)l}{2\pi} + k_2\left(\frac{\pi}{2l}\right)^2\frac{l}{2}\right) \tag{48}$$

$$\omega_n^2 = \frac{EI}{\rho A l^4}\left(\frac{k_1 l^4}{EI} + \left(\frac{\pi}{3\pi-8}\right)\left(\frac{\pi^4}{16} + \frac{k_2 l^2\pi^2}{4EI}\right)\right) \tag{49}$$

$$\omega_n = \sqrt{\frac{EI}{\rho A}\frac{\lambda_1^2}{l^2}} \tag{50}$$

$$\lambda_1^2 = \left(\alpha_1 l^4 + \left(\frac{\pi}{3\pi-8}\right)\left(\frac{\pi^4}{16} + \frac{\alpha_2 l^2\pi^2}{4}\right)\right)^{1/2} \tag{51}$$

Table 3: Fundamental frequency parameters λ_1 of cantilever SBo2PEF

Foundation parameters		**Present**	**Rao and Raju [8]**	**Naidu and Rao [11]**
$\alpha_1 l^4$	$\frac{\alpha_2 l^2}{\pi^2}$			
0	0	1.9141	1.9141	1.876 (-1.00%)
0	0.5	2.5191	2.5191	2.496 (-0.92%)
0	1.0	2.8623	2.8623	2.832 (-1.06%)
0	2.5	3.4859	3.4859	3.421 (-1.86%)
1	0	1.9488	1.9488	1.902 (-2.40%)
1	0.5	2.5346	2.5346	2.513 (-0.85%)
1	1.0	2.8729	2.8729	2.869 (-0.14%)
1	2.5	3.4918	3.4918	3.473 (-0.54%)
100	0	3.2634	3.2634	3.202 (-1.88%)
100	0.5	3.4415	3.4415	3.432 (-0.28%)
100	1.0	3.5955	3.5955	3.487 (-3.02%)

100	2.5	4.0448	4.0448	4.002 (-1.06%)
1000	0	10.003	10.003	10.001 (-0.02%)
1000	0.5	10.010	10.010	10.011 (0.1%)
1000	1.0	10.017	10.017	10.017 (0%)
1000	2.5	10.039	10.039	10.041 (0.02%)

The numbers in brackets are differences between the results by [11] and present results.

Equation (59) is utilized for computing least natural frequency parameter λ_1 for cantilever SBo2PEF for

$$\alpha_1 l^4 = 0, 1, 100, 1000, \text{ and } \frac{\alpha_2 l^2}{\pi^2} = 0, 0.5, 1, 2.5;$$

and presented in Table 3. Table 3 also presents λ_1 for the similar problem as computed by Rao and Raju [8] and Naidu and Rao [11]. Table 3 confirms that the least frequencies are identical with results by Rao and Raju [8] and different from Naidu and Rao [11] solutions by various percentages ranging from 0% to -3.02% for various values of foundation parameters

4.0 CONCLUSION

This study has utilized Ritz variational method for the harmonic vibration solutions of SBo2PEF.

In conclusion,

(i) The total energy expression (E_t) for the RVM was developed as the sum of strain energies of the beam, 2PEF and kinetic energy.

(ii) E_t was found to be a functional depending upon the modal displacement $W(x)$ and its spatial derivatives $W'(x)$, $W''(x)$.

(iii) $W(x)$ was found as a finite combination of the shape functions $\varphi(x)$, and E_t becomes a functional in terms of $\varphi(x)$ and generalized parameters c_n.

(iv) The criteria for minimum of E_t yields the frequency equation.

(v) For simply supported SBo2PEF, the exact sinusoidal shape function used gave the exact eigenfrequencies of vibration for any mode of vibration, as confirmed by previous results.

(vi) For clamped-clamped SBo2PEF, the one-parameter shape function gave accurate results for fundamental frequency which were identical with previous results

(vii) For cantilever SBo2PEF, a one-parameter shape function gave accurate fundamental frequency solutions that agreed with past solutions.

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