NULL-CONTROLABILITY CRITERIA FOR NONLINEAR CONTROL SYSTEM

by

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ABSTRACT:
This paper is concerned with the establishment of null controllability in Euclidean pace, of a nonlinear control system. Sufficient conditions for a nonlinear system to be steered to a zero target are developed by means of Brouwer's Fixed Point Theorem.
This method involves the prescription of an appropriate transformation on the solution of the system, which in the first place is assumed to exist. An appropriate mapping of a suitable subset of the Euclidean space into itself is given. Subsequently, this mapping is proved to be continuous. Thus on the application of Brouwer's Fixed Point Theorem it is proved that the nonlinear control system
$x(t) = A(t)x(t) + f(t ,x(t), u(t), x(o) = X_o$
is Euclidean null controllable.

1. INTRODUCTION:

In this paper, we establish the conditions under which the non-linear control system ... $x(t) = A(t)x(t) + f'(t) ,x(t), u(t), x(o) = x_o$ ..-(1.1) is Euclidean null controllable.

Let $E = (-\infty \ ,\infty)$ be the real line, E^n is the n-dimensional Euclidean space and $E^+ = (0, \infty) = I$, say, an interval of the real line. In (1.1) above, x t E^n, A(t) is an .nxn matrix whose elements are continuous functions on E^+, and whose components are summable over suitable subintervals of E^+. We assume that vector f is n-dimensional and that $(f:0,\infty)\ xE^nxE^m \rightarrow E^n$ is continuous and nonlinear in Sx and Lipschitzian in both x and u. The solution of (1.1) is assumed to exist. The control, u, is an m-vector valued measurable function $u: (0, \infty)) \rightarrow \Omega$ forced to lie in a convex, compact nonvoid subset Ω of the m-dimensional Euclidean space E^m. Such a measurable control function u is said to be admissible. By convention, we shall denote the absolutely continuous solution of the control system by $x(t,u,x_o) = x(t)$, say for convenience.

Perhaps, it is relevant here to point out that the problem of forcing the solutions of control systems to zero (ie null controllability) has received a serious attention from researchers allover the world. (see, for instance, Chukwu[1], Schmitendrof and Barmish[2]. This is because the problem is mainly concerned with the solution of practical problems confronting man. As examples'- suppose in (1.1) above that x is the number of people in a community who refuse to pay their tax, or x can be the cause of water shortage in a town, or even x may be the number of cows dying monthly from the rinderpest disease in each case x may be related to certain recognizable local conditions. A and f such that an equation of the form (1.1) is satisfied - we are aware of the advantages and urgency of reducing each of these XIS to zero in finite time, that is, a question of null controllability. This paper, however, adds a new dimension to the solution of the problem since to our knowledge this is the first time that Brouwerts Fixed Point Theorem is made the crucial instrument for deriving the null controllability criterion.

This paper is divided into three major sections. In the first section which is introductory we define the problem and introduce the method of its solution. In the second section - we define the main terminologies relevant to the problem. In it we also state Brouwer's Fixed Point Theorem as applicable in our present context. Here also we establish another fact in the form of a lemma which is crucial for the pecularity of this paper. In the third and final section the main result is stated and proved.

2. PRELIMINARIES:

If $x(t,u,x\ )$ is the continuous solution of the nonlinear control system (1.1), then by the method of variation - of - constants, we have that $x(t,u,x_o)$ is given by

$$x(t,u,x_o) = X(t)x_o + X(t)\int_o^t X^{-1}(s)f(x,x(s),(s))\ ds \qquad (2.1)$$

where $X(t)$, the fundamental matrix satisfies the equation

$X(t) = A(t)X(t)$ (2. 2)

Suppose x is a given solution of the control system (1.1), then null controllabiliy results if we impose the boundary condition:

$$Tx = 0$$

where T is a bounded linear operator defined on $\xi(E^+, E^n$, the space of all bounded and continuous E^n - valued linear operators defined on E^+.

Definition 2.1

The control system (1.1) is said to be Euclidean controllable if for each X_o ε E^n and each X_1 ε E^n there exists a finite time t_1 $(0 < t_1 < \infty)$ and an admissible control $u: (0,\infty) \rightarrow \Omega$ such that the solution $x(t,u,x_o) = x(t)$, say, of (1.1) satisfies $x(o) = X_o$ and $x(t_1) = X_1$.

The system (1.1) is Euclidean null controllable if in the definition 2.1 above we have $X_1 = o$.

Definition 2.2

Let B C E^n. A mapping $f: B \rightarrow E^n$ is said to be topological if it is a diffeomorphism.

Let. $\sigma^n = \{(X_1,x_2, \ldots X_n) : (x12+X2\,2+\ldots+xn2)^{\frac{1}{2}} \leq 1\}$ be the solid unit so here in E^n.

We now state Brouwer's Fixed Point Theorem.

Lerrma 2. 1 (BROUWER'S FIXED POINT THEOREM).

Let $B = f(\sigma^n)$ where f is a topological mapping. Let μ be a continuous mapping of B into itself, i.e. $\mu(B)CB$. Then there exists an x ε B such that $\mu\ (x) = x$.

Note: (i) The f in definition 2.2 and Lemna 2.1 above are arbitrary and should be different from f of the system (1.1):

(ii) Since the existence of x satisfying (1.1) is assumed here, our task is to prove the existence of an x satisfying both (1.1) and (2.3) - this implies the required null controllability condition.

Thus we shall be concerned with the existence of a fixed point which implies, by Brouwer's Fixed Point Theorem, the existence of a solution to the problem (1.1), (2.3) provided a suitable $\underline{\mu}$ is defined on B, and $\underline{\mu}$ is shown to be continuous.

Now, let ξ be the Banach space of all bounded continuous functions from E^{+} to E^{n} defined earlier and suppose it has the sup norm $\|.\|$ that is if $\overset{o}{\sigma}\, \varepsilon\, \xi$ then $\|g\| = \sup|g(t)|$.

In E^n, let $\|x\| = \overset{n}{\Sigma} |X_i|$

let μ be a positive integer and S_μ be defined as follows:-

$S_\mu = \{x\ \varepsilon\ E^n : \|x\| \leq \mu\}$

With X(t) the fundamental matrix solution of (1.1) let us denote as in the method used by Kartsatos[3] for boundary value problems $T_1X = \{T\ x(t_1), Tx(t_2), \ldots\}$, that is, T_1X is the matrix whose columns are the values of T at the corresponding columns of X. Under this condition, for any $\alpha\ \varepsilon\ E^n$.

We assume that the matrix $T_1 X$ has an inverse. We now have the following crucial lemna,

lenma 2. 2.: Consider the linear control system

$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ x(o) = x_o^1,$ (2.5)

B is an nxm matrix of continuous functions, assume that the solution x(t), say, of (2.5) exists and that the matrix T_1X is invertible. Then the solution of (2.5 satisfying the boundary condition (2.3) is

$x(t) = p(t) - x(t)\ T_1^{-1}p$ (2.6)

where

$$p(t) = \int_o^t x(t)x^{-1}(s)B(s)\,u(s)\,ds \qquad (2.7)$$

and the corresponding initial condition X_0^1 is

$x_o^1 = T_1X\ {-1}\ Tp$ (2.8)

The above formulae (2.6) and (2.8) are for linear systems. A carry-over to nonlinear systems is straight forward as applied in the main result below.

Proof of Lemna 2. 2:

The variation-of-constants formula gives the solution of (2.5) as

$$x(t) = x(t)x_0^1 + x(t)x\int_o^t x(t)x^{-1}(s)B(s)\,u(s)\,ds, \qquad (2.9)$$

that is $x(t) = X(t)x_O^1 + p(t)$

where p(t) is as defined in (2.7) above

Applying T to this we have

$T_x(t) = T_x(t)x_o^1 + T_P(t)$

Consequently, we have $T_X = TXx_o\ 1 + T_P$.

But from (2.3) $T_X = 0$, we have for x satisfying (2 5).

$O = TXx_O^1 + T_P$, or $TXx_O^1 = -T_P$.

But from (2.4) we have $T_1X\ x_O\ 1 = TXx_O\ 1 = -T_P$, and as T_1X is invertible, we have

$x_o^1 = -\ T_1X^{-1}\ T_P$ as required in (2.8) above. Substituting this value of x_o^1 (i.e. the initial point of (2.5) satisfying the boundary condition (2.3) into (2.9) and taking into account the definition (2.7) for P(t)), we get the required result (2.6). This completes the proof.

3. MAIN RESULT:

Theorem 3.1

Suppose $x(t,x_o,u)$ is the continuous solution of the nonlinear control system

$$\dot{X}(t) = A(t)x(t) + f(t)x(f),u(t)), \qquad (1.1)$$

with the initial condition $x(o) = x_o$ of the form (2.8) in lemma 2.2 given by

$X_o = T_1X - 1T_r$ where r(t) is defined as (3.1)

$$r(t) = \int_0^t X(t)X^{-1}(s)f(s,x\{s),u(s)\ ds. \qquad (3.2)$$

We assume the existence and uniqueness of the solution x(t,:xo,u) of (1.1) with the initial point Xo ε S_μ. Let there exist a constant c > 0 such that

$$\|x(t, x_O, u)\| \le o$$

Whenever sup $\|X-1(t)f(t, x(t), u(t))\| = q(t)$, we have

$$N = \int_0^\infty q(s)\,ds < \infty$$

Then if $L = \max \|x(t)\|$ and

$t \ge o$

$$LN \left\|T_1 X^{-1}\right\| \|T\| \le \mu$$

then there exist at least one solution of (1.1) above satisfying the boundary condition (2.3). That is, the system (1.1) is Euclidean null controllable.

Proof:

Let $\underline{\mu}$ $S_\mu \to \infty$ be the function defined by

$$\underline{\mu} = \|T_1X\|^{-1}T \int_0^t X(t);X^{-1}(s)f(f, x(s), x(s))\ ds. \qquad (3.3)$$

From the definition of S_μ and the hypothesis of the theorem $\underline{\mu}$ is a mapping of S_μ into itself. We prove continuous. Let

$\{x_n\}$ ε S_μ, x^* ε S_μ, with $\lim_{n\to\infty} \|x_n - x^*\| = o$. Then from (3.3)

We have

$$\|\underline{\mu}x_n - ux^*\| = \left\|[T_1X]^{-1}(T \int_0^t X(t)X^{-1}(s)\left[f(s, x_n, u) - f(s, x^*, u)\right]ds\right\|$$

Then, we have

$$\left\| \underline{\mu} x_n - ux^* \right\| \le \left\| T_1 X^{-1} \right\| \left\| T \right\| \int_o^t {}^{-1}(s)\left[f(s, x_n, u) - f(s, x^*, u) \right] dx \tag{3.4}$$

(3.4) …..

In (3.4) above the integrand tends pointwise to zero as n $\to \infty$, and is bounded by 2q (t) which is itself an integrable function, then from (3.4)

$$\left\| \mu x_n - \mu x^* \right\| \le \left\| [T_1 X]^{-1} \right\| \left\| T \right\| LN = LN \left\| [T_1 X]^{-1} \right\| \le \mu$$ from hypothesis.

Hence from Lebesgue Theorem on Dominated Convergence, we have

$\lim_{n \to \infty} \left\| ux_n - ux^* \right\| = o,$ that is, $\underline{\mu}$ is continuous

Then as $\underline{\mu}$ is a continuous mapping of S_μ into itself, Brouwer's Fixed Point Theorem (Lemma 2.1) implies the existence of at least one vector X_o such that

$x_o = [T_1 X]^{-1} T \int_o^t X(t)\, X^{-1}$ (s) f(s, x_o, u) ds, that is $\underline{\mu} x_o = x_o$.

This means that there exists an initial point of the control system (1.1) satisfying the boundary condition (2.3). And so there exists at least one solution x of (1.1) corresponding to x_o such that $\mu x = x$. This means that the given control system (1.1) is Euclidean null controllable. This concludes the proof of the theorem.

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